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FINITE ELEMENT APPROXIMATION OF A NONLINEAR DIFFUSION PROBLEM

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Abstract—This paper deals with the finite element approximation of the nonlinear diffusion problem: $-\text{div}\,(|\text{grad}\,u|^{p-2}\,\text{grad}\,u) = f$. Glowinski and Marrocco[3] have been shown that the rate of convergence decreases as p increases. In this paper we show that the rate of convergence is optimal and independent of p. This theoretical result agrees with the numerical experiments reported in the last section.

1. INTRODUCTION

This paper deals with the nonlinear boundary value problem

$$-\nabla\cdot(\kappa\nabla\varphi) = f(\mathbf{x}) \quad \text{for} \quad \mathbf{x}\in\Omega \tag{1}$$

$$\varphi(\mathbf{x}) = 0 \quad \text{for} \quad \mathbf{x}\in\Gamma \tag{2}$$

defined in an open subset Ω of $\mathbf{R}^n$. The diffusion constant κ is assumed to have the form

$$\kappa = |\nabla\varphi|^{p-2}, \tag{3}$$

thus the problem is linear when $p = 2$ and nonlinear for other values of p.

Also observe that the problem 'degenerates' on subsets of Ω where the solution $\varphi = \varphi_0$ is constant. However, this rarely occurs in physical realizations of the model (1)–(2); indeed, one typically has

$$|\nabla\varphi_0| \geq \sigma_0 \tag{4}$$

for some $\sigma_0 > 0$. See for example[1] and[2].

Glowinski and Marrocco[3] have considered finite element approximations to (1)–(2). Using energy arguments (i.e. monotone operator theory) similar to those employed by Strauss[4], they were able to obtain *a priori* error estimates for the approximation. The striking thing about these estimates is that they predict a *serious reduction* in the *rate* of *convergence* as the number *p increases* in (3). In particular, if one uses a finite element space based on piecewise polynomial functions of degree $k-1$ on a grid with spacing $h > 0$, Glowinski and Marrocco showed that the error

$$\|\varphi_0-\varphi_h\|_{1,p} \equiv \left\{\int_\Omega [|\nabla(\varphi_0-\varphi_h)|^p + |\varphi_0-\varphi_h|^p]\right\}^{1/p} \tag{5}$$

in the finite element approximation is proportional to

$$h^{(k-1)/(p-1)}. \tag{6}$$

Since we know (see e.g. [5]) that there is some function $\tilde{\varphi}_h$ in the finite element space for which

$$\|\varphi_0-\tilde{\varphi}_h\|_{1,p} \leq (\text{constant})h^{k-1}, \tag{7}$$

(6) suggests that the method produces optimal approximations only for linear problems ($p = 2$),

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and that accuracy of the method is seriously affected as the degree p of the nonlinearity is increased.

The question of the sharpness of (6) was left opened in [3], and the starting point of the present authors' study was to investigate the actual rates of convergence in selected numerical experiments. The latter, several of which are reported in Section 4, indicated that for *smooth solutions* (6) was anything but sharp, and the better rate (7) was actually obtained. The experiments included solutions φ for which (4) was valid, and solutions where $\nabla\varphi_0$ vanished on sets of measure zero.

The major theoretical result of this paper is an error bound in appropriate Sobolev norms

$$\|v\|_{m,r} = \left\{\int_\Omega \sum_{|\alpha|\le m} |D^\alpha v|^r\right\}^{1/r} \tag{8}$$

which shows that optimal rates and not analogs of (6) are obtained. It is to be emphasized that the ability to approximate (i.e. the existence of functions $\tilde{\varphi}_h$ satisfying inequalities like (7)) is central to our analysis. Thus if the solution φ_0 does have singularities, our statement that optimal rates of convergence are obtained is valid only if appropriate singular functions are included in the approximation (see e.g. [6]).

2. VARIATIONAL FORMULATION

The standard Galerkin procedure that was used in [3] and in this paper is based on the form

$$a(u|v, w) = \int_\Omega |\nabla u|^{p-2}\nabla v \cdot \nabla w \tag{1}$$

defined for u, v, w in the linear space H consisting of functions u satisfying

$$\|u\|_{1,p} < \infty \tag{2}$$

$$u = 0 \quad \text{on} \quad \Gamma. \tag{3}$$

Problem VP. Find a φ_0 in H such that

$$a(\varphi_0|\varphi_0, \psi) = \langle\psi, f\rangle \tag{4}$$

for all $\psi \in H$, where

$$\langle\psi, f\rangle = \int_\Omega \psi f. \tag{5}$$

The existence and uniqueness of $\varphi_0 \in H$ satisfying (4) for a given f,

$$\|f\|_{0,2} \equiv \left(\int_\Omega |f|^2\right)^{1/2} < \infty \tag{6}$$

has been proved by Strauss [4].

To approximate the solution φ_0 of (4) we introduce a family of finite dimensional subspaces $\{S_h\}$ of H parameterized by $h > 0$. If N_h denotes the dimension of S_h we put

$$h = (1/N_h)^{1/n}. \tag{7}$$

This convention is convenient for applications to finite elements in that the h defined by (7) can be regarded as an average mesh spacing. In addition, the following assumption concerning approximability in S_h can be readily verified for finite element spaces with h defined by (7) (see e.g. [5]).

Approximation property

There is an integer $k \geqq 2$ and numbers $C_0{}^q$, $C_1{}^q$ depending on q, $1 \leq q \leq \infty$, but independent of h such that for any $v \in H$ there exists a

$$\tilde{v}_h \in S_h \tag{8}$$

satisfying

$$\|v - \tilde{v}_h\|_{l,q} \leqq C_l^q h^{t-l} \|v\|_{t,q} \tag{9}$$

for $0 \leq l \leq 1$ and $l < t \leq k$.

The approximation $\varphi_h \in S_h$ to φ_0 is defined by the following variational analog of (4).

Problem AVP. Find a $\varphi_h \in S_h$ such that

$$a(\varphi_h | \varphi_h, \psi^h) = \langle \psi^h, f \rangle \tag{10}$$

for all ψ^h in S_h.

Once a basis has been selected for S_h, (10) is equivalent to a set of N_h nonlinear equations. This is verified in[3], who in fact established the following result.

THEOREM (Glowinski and Marrocco). Let

$$2 \leq p < \infty. \tag{11}$$

Then the problem AVP has a unique solution $\varphi_h \in S_h$. Moreover if the approximation property holds, there is a positive number $C = C(C_0, C_1, p)$ for which

$$\|\varphi_0 - \varphi_h\|_{1,p} \leqq C h^{(k-1)/(p-1)} \|\nabla \varphi_0\|_{0,p}^{(p-2)/(p-1)} \|\varphi_0\|_{k,p}^{1/(p-1)}. \tag{12}$$

A similar result was also proved for the case $1 \leq p < 2$. It is perhaps of interest to note that (12) is an easy consequence of the inequalities

$$a(u|u, u-v) - a(v|v, u-v) \geq \alpha_p \|\nabla(u-v)\|_{0,p}^p \tag{13}$$

$$|a(u|u, w) - a(v|v, w)| \leqq \beta_p \|\nabla(u-v)\|_{0,p} (\|\nabla u\|_{0,p} + \|\nabla v\|_{0,p})^{p-2} \|\nabla w\|_{0,p}, \tag{14}$$

which in turn are direct consequences of

$$(|\mathbf{a}|^{p-2}\mathbf{a} - |\mathbf{b}|^{p-2}\mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \geqq \alpha_p |\mathbf{a} - \mathbf{b}|^p \tag{15}$$

$$||\mathbf{a}|^{p-2}\mathbf{a} - |\mathbf{b}|^{p-2}\mathbf{b}| \leqq \beta_p |\mathbf{a} - \mathbf{b}| (|\mathbf{a}| + |\mathbf{b}|)^{p-2} \tag{16}$$

where α_p, β_p are positive numbers which depend only on p and $\mathbf{a}$, $\mathbf{b}$ are any vectors on $\mathbf{R}^n$.

Indeed, to show that (13)–(14) imply (12) note that (4) and (10) can be combined to give

$$a(\varphi_h | \varphi_h, \psi^h) = a(\varphi_0 | \varphi_0, \psi^h) \tag{17}$$

for any ψ^h in S_h. Thus if $\hat{\varphi}_h$ is *any* function in S_h

$$\begin{aligned} &a(\varphi_0|\varphi_0, \varphi_h - \hat{\varphi}_h) - a(\hat{\varphi}_h|\hat{\varphi}_h, \varphi_h - \hat{\varphi}_h) \\ &= a(\varphi_h|\varphi_h, \varphi_h - \hat{\varphi}_h) - a(\hat{\varphi}_h|\hat{\varphi}_h, \varphi_h - \hat{\varphi}_h) \end{aligned} \tag{18}$$

Using the inequality (14) with $u = \varphi_0$, $v = \hat{\varphi}_h$, and $w = \varphi_h - \hat{\varphi}_h$ and the inequality (13) with $u = \varphi_h$, $v = \hat{\varphi}_h$, we see that (18) implies

$$\alpha_p \|\nabla(\varphi_h - \hat{\varphi}_h)\|_{0,p}^{p-1} \leqq \beta_p \|\nabla(\varphi_0 - \hat{\varphi}_h)\|_{0,p} (\|\nabla \varphi_0\|_{0,p} + \|\nabla \hat{\varphi}_h\|_{0,p})^{p-2}. \tag{19}$$

By letting $\hat{\varphi}_h$ be the best approximation in the norm $\|\nabla w\|_{0,p}$ on H (recall the boundary condition (3) is satisfied by any function in H), we can assume

$$\|\nabla \hat{\varphi}_h\|_{0,p} \leqq \|\nabla \varphi_0\|_{0,p} \tag{20}$$

and

$$\|\nabla(\varphi_0 - \hat{\varphi}_h)\|_{0,p} \leqq C_1 h^{k-1} \|\varphi_0\|_{k,p} \tag{21}$$

Combining (19)–(21) with the triangle inequality we obtain (12).

3. IMPROVED ERROR ESTIMATES

This section is devoted to improved error estimates in the L_2 norms $\|\cdot\|_{0,2}$ and $\|\cdot\|_{1,2}$. We shall tacitly assume throughout this section that the approximation property holds.

Instead of (15, Section 2) we shall use the following inequality.

LEMMA 1. There is a positive number γ_p such that

$$(|\mathbf{a}|^{p-2}\mathbf{a} - |\mathbf{b}|^{p-2}\mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \geq \gamma_p (|\mathbf{a}| + |\mathbf{b}|)^{p-2} |\mathbf{a} - \mathbf{b}|^2 \tag{1}$$

A proof of (1) is given in Appendix 1 to this paper. Observe that an estimate for

$$\nabla(\varphi_h - \hat{\varphi}_h)$$

can be obtained directly from (1) and (16, Section 2). Indeed, starting with (18, Section 2) we have

$$\gamma_p \int_\Omega |\nabla \varphi_0|^{p-2} |\nabla(\varphi_h - \hat{\varphi}_h)|^2 \leqq \beta_p \int_\Omega (|\nabla \hat{\varphi}_h| + |\nabla \varphi_0|)^{p-2} |\nabla(\varphi_h - \hat{\varphi}_h)| |\nabla(\varphi_0 - \hat{\varphi}_h)|$$

We now choose $\hat{\varphi}_h$ such that

$$|\nabla \hat{\varphi}_h| \leqq C |\nabla \varphi_0|$$

for some constant C independent of h, φ_0, and $\hat{\varphi}_h$†. Using Schwartz's inequality we have the following theorem.

THEOREM 1. There is a constant C independent of h, φ_0 and φ_h such that

$$\left\{ \int_\Omega |\nabla \varphi_0|^{p-2} |\nabla(\varphi_0 - \varphi_h)|^2 \right\}^{1/2} \leqq C h^{k-1} \|\nabla \varphi_0\|_{0,\infty}^{p-2} \|\varphi_0\|_{k,2} \tag{2}$$

Thus if

$$\sigma_0 \leq |\nabla \varphi_0| \leq \sigma_1 \quad \text{in} \quad \Omega \tag{3}$$

for positive finite numbers σ_0 and σ_1 then

$$\|\varphi_0 - \varphi_h\|_{1,2} \leq C h^{k-1} \|\varphi_0\|_{k,2}, \tag{4}$$

for some new constant $C = C(\sigma_0, \sigma_1)$ which of course is best possible.

†The constant $C = 2$ is obtained by letting $\hat{\varphi}_h$ be the best approximation to φ_0 in the norm

$$\operatorname*{Sup}_{x \in \Omega} |\nabla w(\mathbf{x})|$$

on the Sobolev space $W_0^{1,\infty}(\Omega)$.

We now turn to estimates for the L_2 errors

$$\|\varphi_0-\varphi_h\|_{0,2}=\left\{\int_\Omega|\varphi_0-\varphi_n|^2\right\}^{1/2}.$$

By the mean value theorem we can choose the function

$$\sigma=\sigma(\varphi_0,\varphi_h,p)$$

to satisfy

$$|\nabla\varphi_0|^{p-2}\nabla\varphi_0-|\nabla\varphi_h|^{p-2}\nabla\varphi_h=\sigma\nabla(\varphi_0-\varphi_h). \tag{5}$$

The inequalities (1) and (16, Section 2) imply

$$\gamma_p|\nabla\varphi_0|^{p-2}\le\sigma\le\beta_p(|\nabla\varphi_0|+|\nabla\varphi_h|)^{p-2}.$$

Let

$$a_\sigma(u,v)=\int_\Omega\sigma\nabla u\cdot\nabla v. \tag{6}$$

Following the now classical 'Nietsche trick' (see [5]) we consider the solution w to

$$\begin{aligned}\nabla\sigma\cdot\nabla w&=\varphi_0-\varphi_h \quad \text{in} \quad \Omega\\ w&=0 \qquad\quad\ \ \text{on} \quad \Gamma,\end{aligned}$$

i.e. the function $w\in H$ satisfying

$$a_\sigma(u,w)=\langle u,\varphi_0-\varphi_h\rangle \quad \text{for all} \quad u\in H. \tag{7}$$

Observe that

$$\begin{aligned}\int_\Omega|\varphi_0-\varphi_h|^2&=\langle\varphi_0-\varphi_h,\varphi_0-\varphi_h\rangle\\&=a_\sigma(\varphi_0-\varphi_h,w)\\&=a(\varphi_0|\varphi_0,w)-a(\varphi_h|\varphi_h,w)\end{aligned}$$

(by (5)). Since a $(\cdot|\cdot,\cdot)$ is linear in the third variable and since ((17), Section 2) is valid,

$$\begin{aligned}\int_\Omega|\varphi_0-\varphi_h|^2&=a(\varphi_0|\varphi_0,w-\tilde{w}_h)-a(\varphi_h|\varphi_h,w-\tilde{w}_h\rangle\\&\leqq\beta_p\|\nabla(w-\tilde{w}_h)\|_{0,2}\|\nabla(\varphi_0-\varphi_h)\|_{0,2}\\&\quad\times(\|\nabla\varphi_0\|_{0,\infty}+\|\nabla\varphi_h\|_{0,\infty})^{p-2}.\end{aligned} \tag{8}$$

Since w is a solution of the boundary value problem (7), there is an *a priori* inequality of the form

$$\|w\|_{r,2}\leqq C\|\varphi_0-\varphi_h\|_{0,2} \tag{9}$$

provided σ is a bounded measurable function. The number r is the regularity index of Ω whose precise definition is:

Definition. The *regularity index of* Ω is the largest number r such that there is a constant

$C = C(r, \Omega)$ for which

$$\|u\|_r \leq C\|g\|_0 \tag{10}$$

where ever

$$\Delta u = g \quad \text{in} \quad \Omega \tag{11}$$

$$u = 0 \quad \text{on} \quad \Gamma. \tag{12}$$

It is known that $1 < r \leq 2$ (see for example [7, Ch. 3]) and that $r = 2$ if Ω is convex or for example has a C^2 boundary.

Assuming (9) holds, we choose $\tilde{w}_h$ such that

$$\|\nabla(w - \tilde{w}_h)\|_{0,2} \leq Ch^{r-1}\|w\|_{r,2} \tag{13}$$

Combining this with (8)–(9) we obtain the estimate given in the following theorem.

Theorem 2. Let

$$\sigma_0 \leq |\nabla \varphi_0| \leq \sigma_1 \tag{3'}$$

hold and let

$$|\nabla \varphi_h| \leq \sigma_2 \tag{14}$$

Then there is a constant C depending only on Ω, σ_i ($i = 0, 1, 2$) and the constants in the approximation property such that

$$\|\varphi_0 - \varphi_h\|_{0,2} \leq Ch^{r-1}\|\nabla(\varphi_0 - \varphi_h)\|_{0,2} \tag{15}$$

where r is the regularity index of Ω.

To some extent (15) is less satisfactory than (4) in that we had to in essence make assumptions about the approximation φ_h to obtain (15). In particular, it required that we assume that both (3′) and (14) hold while (4) only used (3′).

4. NUMERICAL RESULTS

Our numerical results deal with the two dimensional case in a unit square with piecewise linear and piecewise quadratic elements. In particular these experiments confirm that the optimal rates of convergence are obtained, i.e.

$$\|\varphi_0 - \varphi_h\|_l = O(h^{2-l}) \tag{1}$$

for linear elements and

$$\|\varphi_0 - \varphi_h\|_l = O(h^{3-l}) \tag{2}$$

for quadratics.

If $\varphi_1^h, \ldots, \varphi_N^h$ is a basis for either of these spaces, then the problem AVP is equivalent to the system of N nonlinear equations

$$K(\mathbf{q})\mathbf{q} = \mathbf{Q} \tag{3}$$

in the N weights $q_1, \ldots, q_N$ in

$$\varphi_h(\mathbf{x}) = \sum_{j=1}^{N} q_j\varphi_j^h(\mathbf{x}).$$

The (i, j) entry of the matrix $K(\mathbf{q})$ is

$$a(\varphi_h|\varphi_j^h, \varphi_i^h),$$

and the i-th entry of $\mathbf{Q}$ is

$$\int_\Omega \varphi_i^h f.$$

We solved the nonlinear system (3) by Newton's method which is equivalent to applying the finite element method at each Newton iteration to the linear problem associated with the bilinear form

$$A(\varphi, \psi) = \int_\Omega |\nabla\xi|^{p-2}\nabla\varphi \cdot \nabla\psi + (p-2)\int_\Omega |\nabla\xi|^{p-4}(\nabla\xi \cdot \nabla\varphi)(\nabla\xi \cdot \nabla\psi). \tag{4}$$

Indeed, if $\mathbf{q}^{(k)}$ is the k-th Newton iterate and if

$$\varphi_h^{(k)}(\mathbf{x}) = \sum_{j=1}^{N} q_j^{(k)}\varphi_j^h(\mathbf{x}),$$

then $\varphi_h^{(k+1)}$ is the solution of the variational problem

$$A(\varphi_h^{(k+1)}, \psi^h) = \int_\Omega \psi^h f + (p-2)\int_\Omega |\nabla\varphi_h^{(k)}|^{p-4}(\nabla\varphi_h^{(k)} \cdot \nabla\varphi_h^{(k)})(\nabla\varphi_h^{(k)} \cdot \nabla\psi^h) \tag{5}$$

for all ψ^h in S_h where ξ in (4) is $\varphi_h^{(k)}$.

We experienced no problems with this iteration, quadratic convergence being obtained for all the starting values we selected.

In our first experiment we consider the case where the exact solution is

$$\varphi_0(x_1, x_2) = \sin(x_1 + x_2) \tag{6}$$

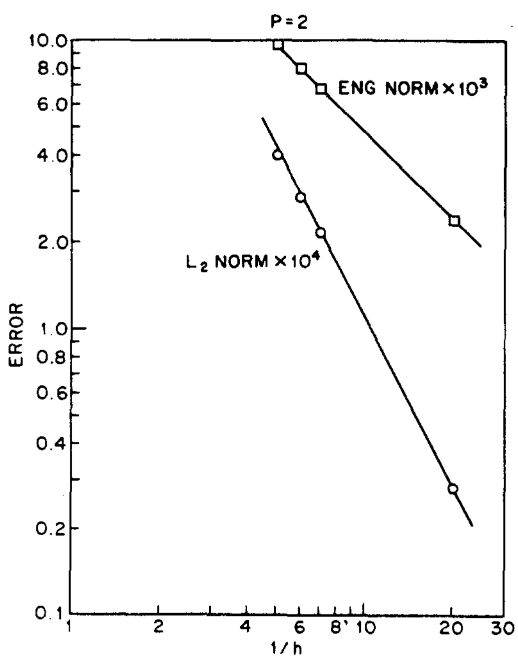

Fig. 1a.

while this does not have homogeneous boundary data, this is not an essential point since the results in the paper are easily generalized to cover nonzero boundary data. In Fig. 1 we plot the L_2 norm of the difference $\varphi_I - \varphi_h$ between the Galerkin approximation φ_h and the interpolant φ_I of (6) for various values of p and for linear elements. Also plotted is the difference $\varphi_I - \varphi_h$ in the 'energy norm'

$$\|\epsilon\|^2_{\mathrm{ENG}} = \int_\Omega |\nabla\varphi_I|^{p-2}\nabla\epsilon \cdot \nabla\epsilon \tag{7}$$

The corresponding plots for quadratic elements are given in Fig. 2.

These results confirm that the correct rates of convergence are given by (2 and 15, Section

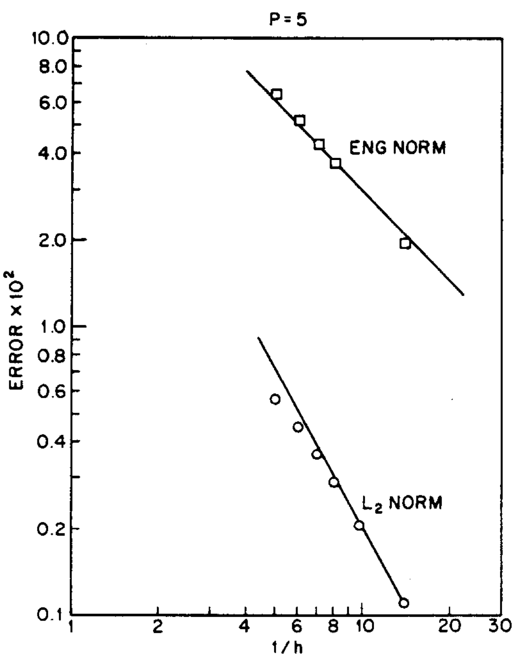

Fig. 1b.

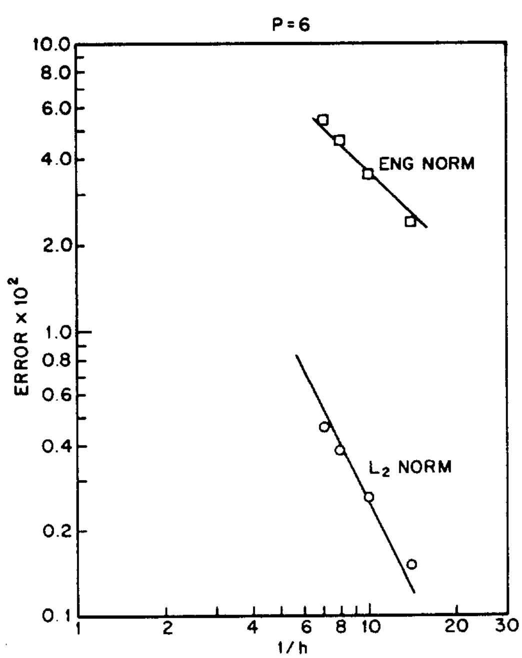

Fig. 1c.

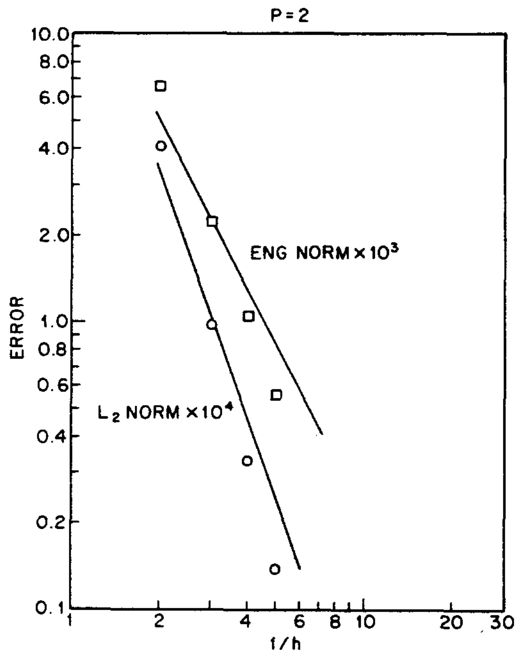

Fig. 2a.

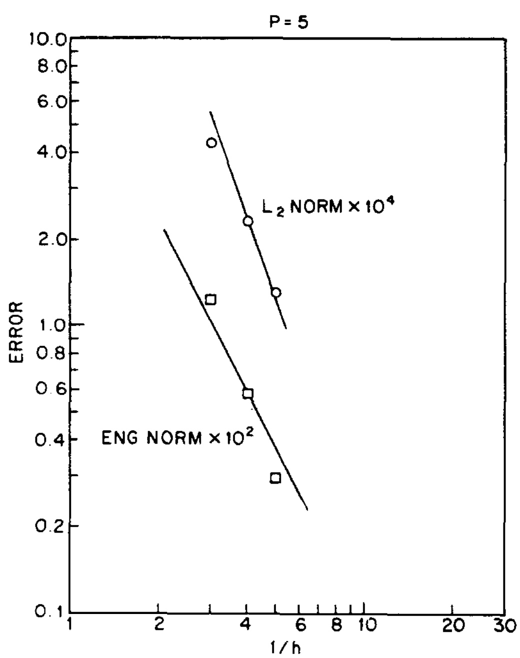

Fig. 2b.

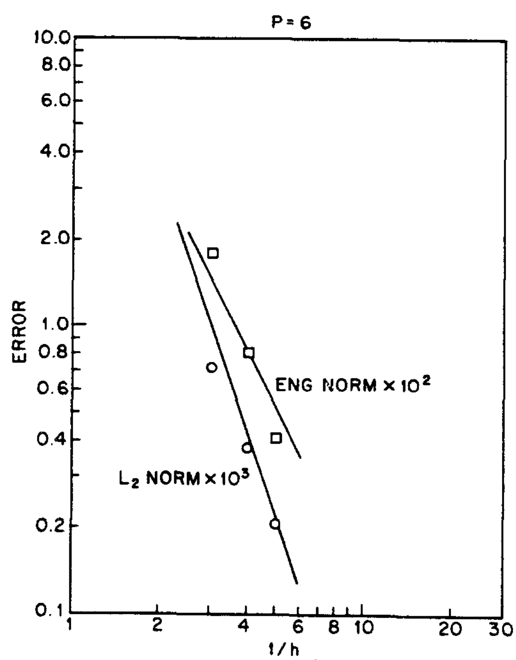

Fig. 2c.

3) and not by (12, Section 2). Our next experiments confirm that the norms in the error estimates are the right ones. For quadratic elements we consider a solution

$$\varphi_0(x_1, x_2) = x_1^{2.55} \tag{8}$$

which satisfies $\|\varphi_0\|_{3,2} < \infty$ and $\|\nabla\varphi\|_{0,\infty} < \infty$ but *not* $\|\varphi_0\|_{3,p} < \infty$. The errors are plotted in Fig. 3.

The second case considered was

$$\varphi_0(x_1, x_2) = x_1^{1.25} \tag{9}$$

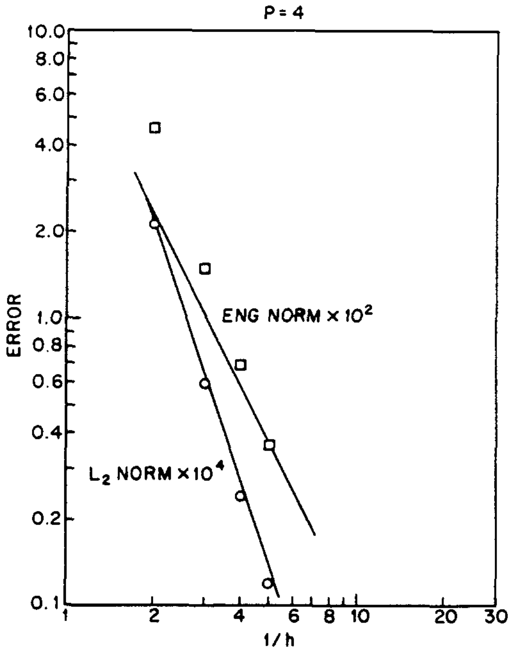

Fig. 3a.

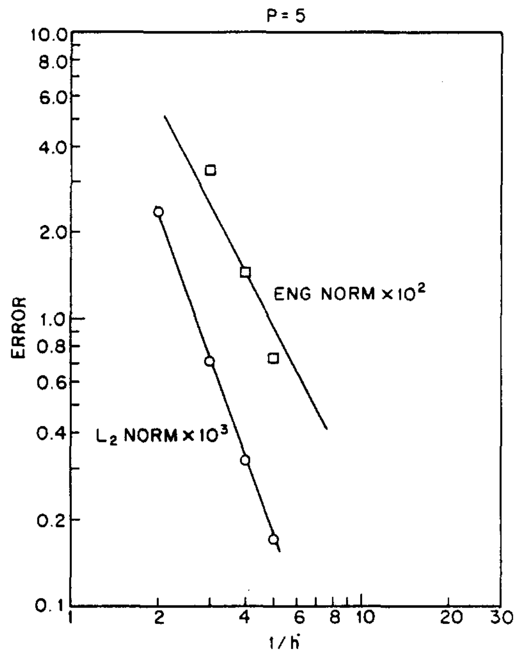

Fig. 3b.

where $\|\varphi_0\|_{3,2}=\infty$ but $\|\varphi_0\|_{1.75-\epsilon,2}<\infty$ for all $\epsilon>0$. Our theory is easily modified to show in this case that

$$\|\varphi_0-\varphi_h\|_{0,2}=O(h^{1.75-\epsilon}) \tag{10}$$

$$\|\varphi_0-\varphi_h\|_{\mathrm{ENG}}=O(h^{0.75-\epsilon}) \tag{11}$$

all $\epsilon>0$. A good model for these rates is

$$\|\varphi_0-\varphi_h\|_{0,2}=O(-h^{\alpha}\ln h) \tag{12}$$

$$\|\varphi_0-\varphi_h\|_{\mathrm{ENG}}=O(-h^{\beta}\ln h) \tag{13}$$

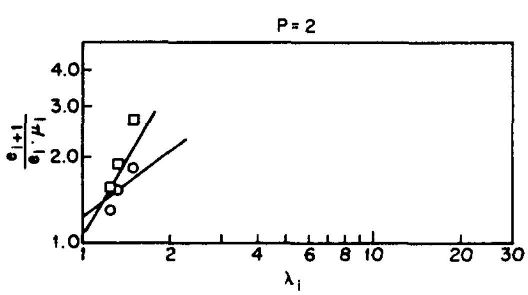

Fig. 4a.

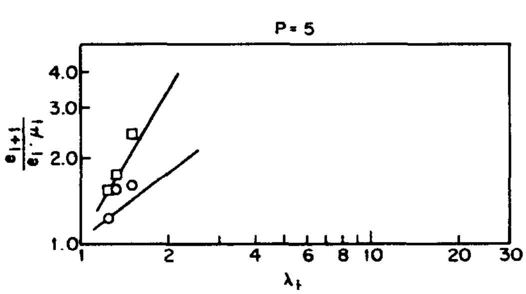

Fig. 4b.

where $\alpha = 1.75$ and $\beta = 0.75$. The numerical results for various values of p are arranged in Tables 1–2. In these tables we denoted $\lambda_i = \ln(h_{i+1}/h_i)$, $\mu_i = 1 + (\ln \lambda_i/\ln h_{i+1})$. The quantities $(e_{i+1}/e_i \cdot \mu_i)$ where plotted vs λ_i on a log–log scale (Fig. 4), so that the slope gives the rate of convergence.

$\varphi_0(x_1, x_2) = x_1^{1.25}$
Quadratic elements
Table 1. $p = 2$

h_i	$e_i = L^2$ NORM	λ_i	μ_i	$\frac{e_{i+1}}{e_i\mu_i}$	α	e_i = ENG NORM	$\frac{e_{i+1}}{e_i\mu_i}$	β_2
1/2	0.00067	3/2	0.6309	2.72	2.47	0.00903	1.83	1.49
1/3	0.00039	4/3	0.7925	1.89	2.22	0.00782	1.53	1.49
1/4	0.00026	5/4	0.8614	1.59	2.07	0.00643	1.29	1.15
1/5	0.00019					0.00578		

Table 2. $p = 5$

h_i	$e_i = L^2$ NORM	λ_i	μ_i	$\frac{e_{i+1}}{e_i\mu_i}$	α	e_i = ENG NORM	$\frac{e_{i+1}}{e_i\mu_i}$	β_2
1/2	0.00149	3/2	0.6309	2.43	2.19	0.02497	1.60	1.16
1/3	0.00097	4/3	0.7925	1.75	1.94	0.02470	1.55	1.53
1/4	0.00070	5/4	0.8614	1.53	1.92	0.02009	1.21	0.88
1/5	0.00053					0.01918		

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APPENDIX

Proof of lemma 1 *in Section* 3

Consider the inequality

$$(|\mathbf{a}|^{p-2}\mathbf{a} - |\mathbf{b}|^{p-2}\mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) \geq \gamma_p(|\mathbf{a}| + |\mathbf{b}|)^{p-2}|\mathbf{a} - \mathbf{b}|^2 \tag{1}$$

Glowinski and Marrocco[3] proved this inequality for $1 < p \leq 2$ and $\mathbf{a}, \mathbf{b} \in \mathbf{R}^2$. We want to verify (1) for other values of p namely $2 < p < \infty$ and $\mathbf{a}, \mathbf{b} \in \mathbf{R}^n$.

If $\mathbf{a} = \mathbf{b}$ then (1) is true for all γ_p. If $\mathbf{a} = 0$ or $\mathbf{b} = 0$ then (1) is valid for $0 < \gamma_p \leq 1$. Suppose now that $\mathbf{a} \neq \mathbf{b}$ and $\mathbf{b} \neq \mathbf{0}$. Let

$$\varphi(\mathbf{a}, \mathbf{b}) = \frac{(|\mathbf{a}|^{p-2}\mathbf{a} - |\mathbf{b}|^{p-2}\mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})}{|\mathbf{a} - \mathbf{b}|^2(|\mathbf{a}| + |\mathbf{b}|)^{p-2}}$$

$\varphi(\mathbf{a}, \mathbf{b})$ is homogeneous thus we can assume $|\mathbf{b}| = 1$, and by rotation of coordinate system $\mathbf{b} = (0, \ldots, 0, 1)$. Clearly, $\varphi(\mathbf{a}, \mathbf{b}) > 0$ for $\mathbf{a} \neq \mathbf{b}$, and $\lim_{|\mathbf{a}| \to +\infty} \varphi(\mathbf{a}, \mathbf{b}) = 1$. The latter follows when noting that the dominating term in both numerator and denominator is $|\mathbf{a}|^p$. Thus it suffices to show that $\liminf_{\mathbf{a} \to \mathbf{b}} \varphi(\mathbf{a}, \mathbf{b}) > 0$. Using polar coordinates we have

$$a_1 = \rho \sin\theta \sin\theta_1 \ldots \sin\theta_{n-2} \sin\theta_{n-1}$$

$$a_2 = \rho \sin\theta \sin\theta_1 \ldots \sin\theta_{n-2} \cos\theta_{n-1}$$

$$a_3 = \rho \sin\theta \sin\theta_1 \ldots \cos\theta_{n-2}$$

$$\ldots$$

$$a_n = \rho \cos\theta + 1$$

$$|\mathbf{a}-\mathbf{b}| = \rho$$
$$|\mathbf{a}| = (1+2\rho\cos\theta+\rho^2)^{1/2}$$
$$(\mathbf{a},\mathbf{b}) = a_n = 1+\rho\cos\theta$$
$$\varphi(\mathbf{a},\mathbf{b}) = \frac{1+[1+2\rho\cos\theta+\rho^2]^{p/2} - (1+\rho\cos\theta)[1+(1+2\rho\cos\theta+\rho^2)^{(p-2)/2}]}{\rho^2[1+(1+2\rho\cos\theta+\rho^2)^{1/2}]^{p-2}}$$

Expanding numerator and denoting by $\epsilon(\epsilon,\theta)$ all terms of order higher than ρ^2, we have

$$\varphi(\mathbf{a},\mathbf{b}) = \frac{\rho^2[1+(p-2)\cos^2\theta]+\epsilon(\rho,\theta)}{\rho^2[1+(1+2\rho\cos\theta+\rho^2)^{1/2}]^{p-2}}$$

If $\mathbf{a}\to\mathbf{b}$ then $\rho\to 0$ and $(\epsilon(\rho,\theta)/\rho^2)\to 0$. Thus

$$\liminf_{\rho\to 0} \varphi(\mathbf{a},\mathbf{b}) = \frac{1+(p-2)\cos^2}{2^{p-2}}\theta > 0.$$